

Lecture 12 (March 7, 2016)

Non autonomous systems $\dot{x} = f(t, x)$

where $f: [0, \infty) \times D \rightarrow \mathbb{R}^n$ is piecewise continuous in t & locally Lipschitz in x on $[0, \infty) \times D$, and $D \subseteq \mathbb{R}^n$ is a domain that contains $x=0$.

Stability definitions

$x=0$ is (stability behavior will in general be dependent on t_0)

□ stable, if for $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon, t_0) > 0$ s.t.

$$\|x(t_0)\| < \delta \rightarrow \|x(t)\| < \varepsilon \quad \forall t \geq t_0 \geq 0 \quad (*)$$

□ uniformly stable, if for $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$, independent of t_0 s.t. (*) holds.

□ asymptotically stable, if it is stable and $\exists c = c(t_0) > 0$ s.t.

$$x(t) \rightarrow 0 \text{ as } t \rightarrow \infty \quad \forall \|x(t_0)\| < c.$$

□ uniformly asymptotically stable, if it is uniformly stable and $\exists c > 0$ independent of t_0 , s.t. $\forall \|x(t_0)\| < c$, $x(t) \rightarrow 0$ as $t \rightarrow \infty$, uniformly in t_0 , i.e. $\forall \eta > 0$, $\exists T = T(\eta)$ s.t.

$$\|x(t)\| < \eta, \quad \forall t \geq t_0 + T(\eta), \quad \forall \|x(t_0)\| < c$$

□ globally uniformly asymptotically stable, if it is uniformly stable, $\delta(\varepsilon)$ satisfies $\lim_{\varepsilon \rightarrow \infty} \delta(\varepsilon) = \infty$, and for each pair $\eta > 0$, $c > 0$, there

is $T = T(\eta, c) > 0$ s.t. $\|x(t)\| < \eta$, $\forall t \geq t_0 + T(\eta, c)$, $\forall \|x(t_0)\| < c$

Example $\dot{x} = \frac{-x}{1+t}$ has solution $x(t) = x(t_0) \frac{1+t_0}{1+t}$

is uniformly stable and asymptotically stable but not u.a.s.

I.e. $T = T(\varepsilon, t_0)$ is not independent of t_0 .

Pick $\eta = 1$ & $x(t_0) = 2$.

$$t_0 = 0, T = 3 \Rightarrow x(t_0 + T) = \frac{1}{2} < 1$$

$$t_0 = 3, T = 3 \Rightarrow x(t_0 + T) = \frac{8}{7} > 1$$

For fixed T , as $t_0 \rightarrow \infty$, $x(t_0 + T) \rightarrow x(t_0) = 2$

so cannot get a constant T s.t. $x(t_0 + T) < 1 \quad \forall t_0$.

Comparison Functions

Def. A continuous function $\alpha: [0, a) \rightarrow [0, \infty)$ belongs to class K if it is strictly increasing and $\alpha(0) = 0$.

Def. It belongs to class K_∞ if $a = \infty$ and $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$.

Def. A continuous function $\beta: [0, a) \times [0, \infty) \rightarrow [0, \infty)$ belongs to class KL if for each fixed s , the mapping $\beta(r, s) \in K$ w.r.t r and for each fixed r , the mapping $\beta(r, s)$ is decreasing w.r.t s and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$.

Example

1) $\alpha(r) = \tan^{-1}(r) \in K \quad (\alpha'(r) = \frac{1}{1+r^2} > 0, \alpha(0) = 0)$

$\notin K_\infty$ because $\lim_{r \rightarrow \infty} \alpha(r) = \frac{\pi}{2} < \infty$

2) $\alpha(r) = r^c, c > 0 \in K_\infty$

3) $\beta(r, s) = \frac{r}{ksr+1}, k > 0 \in KL$

4) $\beta(r, s) = r^c e^{-s}, c > 0 \in KL$

Connections to Lyapunov analysis

Lemma 4.4. $\dot{y} = -\alpha(y)$, $y(t_0) = y_0$ (scalar)

with α locally Lipschitz, class K function defined on $[0, a]$. For all $0 \leq y_0 < a$, the equation has a unique solution $y(t)$ defined for all $t \geq t_0$ and $y(t) = \varsigma(y_0, t - t_0)$ where ς is a class KL function defined on $[0, a) \times [0, \infty)$.

Example

$$\dot{y} = -ky, \quad ky \in K \text{ for } y \in [0, a)$$

$$y(t) = y_0 e^{-k(t-t_0)} = \varsigma(y_0, t-t_0) \Rightarrow \varsigma(r, s) = r e^{-ks} \in KL$$

Lemma 4.3. Let $V: D \rightarrow \mathbb{R}$ be a continuous, positive definite function, $D \subset \mathbb{R}^n$ contains origin. Let $B_r \subset D$ for some $r > 0$. Then \exists class K functions α_1, α_2 , defined on $[0, r]$, st.

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \quad \forall x \in B_r.$$

If $D = \mathbb{R}^n$, α_1, α_2 are defined on $[0, \infty)$ and inequality holds for all $x \in \mathbb{R}^n$.

If $V(x)$ is radially unbounded, α_1, α_2 can be chosen to belong to class K_∞ .

$$\text{Example. } V = x^T P x \Rightarrow \alpha_1(\|x\|) = \lambda_{\min}(P) \|x\|^2$$

$$\alpha_2(\|x\|) = \lambda_{\max}(P) \|x\|^2$$

$$\lambda_{\min}(P) \|x\|^2 \leq x^T P x \leq \lambda_{\max}(P) \|x\|^2$$

Equivalent notion of stability

Lemma 4.5. $x=0$ is

1) uniformly stable iff \exists class K function α and $c > 0$, independent of t_0 , s.t.

$$\|x(t)\| \leq \alpha(\|x(t_0)\|) \quad \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c$$

2) uniformly asymptotically stable iff \exists class KL function β and $c > 0$, independent of t_0 , s.t.

$$(**) \quad \|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \quad \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c$$

3) globally uniformly asymptotically stable iff $(**)$ holds for any initial state $x(t_0)$.

Uniform stability Theorem

Theorem 4.8. $x=0$ is eq.pt. of $\dot{x} = f(t, x)$ and $D \subset \mathbb{R}^n$ contains $x=0$. Let $V : [0, \infty) \times D \rightarrow \mathbb{R}$ be C^1 s.t.

$$W_1(x) \leq V(t, x) \leq W_2(x)$$

$$\dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0$$

$\forall t \geq 0, \forall x \in D$, where $W_1(x), W_2(x)$ are continuous positive definite functions on D . Then $x=0$ is uniformly stable.

Proof $\dot{V}(t, x) \leq 0$. Choose $r > 0, c > 0$ s.t. $B_r \subset D$ & $c < \min_{x \in D} W_1(x)$.

Then $\mathcal{C}_1 = \{x \in B_r \mid W_1(x) \leq c\}$ is in interior of B_r .

Define time-dependent set $\Omega_{t,c} = \{x \in B_r \mid V(t, x) \leq c\}$

since $W_2(x) \leq c \Rightarrow V(t, x) \leq c$, the set $\mathcal{C}_2 = \{x \in B_r \mid W_2(x) \leq c\} \subset \Omega_{t,c}$.

Also, since $V(t, x) \leq c \Rightarrow W_1(x) \leq c$, $\Omega_{t,c} \subset \mathcal{C}_1$.

Thus for all $t \geq 0$, we have: $\mathcal{C}_2 \subset \Omega_{t,c} \subset \mathcal{C}_1$

Since $\dot{V} \leq 0$ on D , for any $t_0 \geq 0$ & $x_0 \in \Omega_{t_0,c}$, the solution starting at (t_0, x_0) stays in $\Omega_{t,c}$, $\forall t \geq t_0$.

\therefore any solution starting in \mathcal{E}_2 stays in $\Omega_{t_0, c} \subset \mathcal{E}_1$, $\forall t \geq t_0$.

So solution is bounded and defined $\forall t \geq t_0$.

Also, $V(t, x(t)) \leq V(t_0, x(t_0))$, $\forall t \geq t_0$.

By Lemma 4.3. \exists class K functions α_1, α_2 defined on $[0, r]$ s.t.

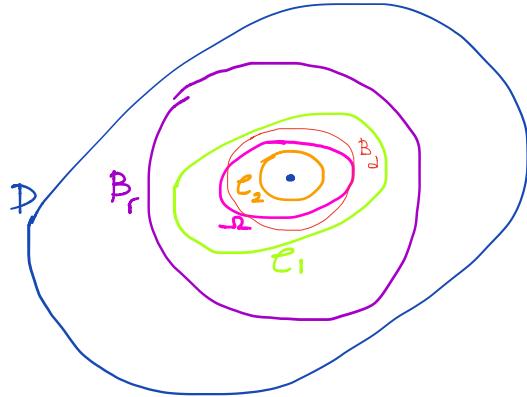
$$\alpha_1(\|x\|) \leq W_1(x) \leq V(t, x) \leq W_2(x) \leq \alpha_2(\|x\|)$$

$$\text{So } \|x(t)\| \leq \alpha_1^{-1}(V(t, x)) \leq \alpha_1^{-1}(V(t_0, x(t_0))) \leq \alpha_1^{-1}(\alpha_2(\|x(t_0)\|))$$

$$\text{Can show that } \alpha_1^{-1}\alpha_2 = \alpha_3 \in \mathcal{K}, \text{ then } \|x(t)\| \leq \alpha_3(\|x(t_0)\|)$$

$$\forall t \geq t_0, \forall x(t_0) \in \Omega_{t_0, c} \Rightarrow \forall x(t_0) \in \mathcal{E}_1 \Rightarrow \forall \|x(t)\| < d \text{ where}$$

$B_d \subset \mathcal{E}_1$. Hence $x=0$ is unstable.



Theorem 4.9. Suppose assumptions of Theorem 4.8 are satisfied and $\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x)$, $\forall t \geq 0$ & $\forall x \in D$ where $W_3(x)$ is a continuous, positive definite on D . Then $x=0$ is u.a.s.

Moreover, if r, c chosen st. $B_r \subset D$ & $c < \min_{\|x\|=r} W_1(x)$, then every trajectory starting in $\mathcal{E}_2 = \{x \in B_r \mid W_2(x) \leq c\}$ satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall t \geq t_0 \geq 0$$

$\beta \in \mathcal{KL}$. If $D = \mathbb{R}^n$ and W_1 is radially unbounded $\Rightarrow x=0$ is g.u.a.s.